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Nonlinear free transverse vibrations of in-plane moving plates: Without and with internal resonances

You-Qi Tang [a], Li-Qun Chen [a,b,*]

[a] Shanghai Institute of Applied Mathematics and Mechanics, Shanghai 200072, China
[b] Department of Mechanics, Shanghai University, Shanghai 200436, China

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ABSTRACT

In this paper, nonlinear free transverse vibrations of in-plane moving plates subjected to plane stresses are investigated. The Hamilton principle is applied to derive the governing equation and the associated boundary conditions. The method of multiple scales is employed to analyze the nonlinear partial differential equation. The solvability conditions are established in the cases without internal resonance and with 3:1 or 1:1 internal resonances. Some numerical examples are presented to demonstrate the effects of in-plane moving speeds on the frequencies. The nonlinear frequencies of the in-plane moving plate without internal resonances are numerically calculated. The relationship between the nonlinear frequencies and the initial amplitudes is showed at different in-plane moving speeds and the nonlinear coefficients, respectively. It is feasible to investigate resonances without the modes not involved in the resonances. The effects of the related parameters are demonstrated for the case of 3:1 and 1:1 internal resonances, respectively. The differential quadrature scheme is developed to solve numerically the governing equation and confirm results via the method of multiple scales.

1. Introduction

Axially moving systems are present in various industrial applications such as the paper and plastic sheets in process, the steel strip in a thin steel sheet production line, conveyor belts and chain in power transmission lines, aerial cableways, band the cable, the band saw blade, and the like. To ensure that the systems are under stable operation, transverse vibrations of these devices have been investigated to avoid the possible resulting fatigue and low quality. Therefore, the vibration of such systems is necessary for the analysis and design of the broad class of technological devices.

A lot of earlier works for modeling two-dimensional axially moving systems used the one-dimensional string or beam theory instead of the plate theory to avoid the complication. This simplification leads to useful and reasonable results when the axially moving systems were narrow, the effects of the boundary conditions at the width ends on the prediction are negligible, or the axially moving systems had no stress variations across the width, but two-dimensional analysis is required for the modeling of many problems such as wide width plates, no free lateral boundaries, various forces across the width, catching higher modes of vibration, intermediate supports or buckling, etc.

There have been so many publications on axially moving one-dimensional structures such as strings and beams, but works on in-plane moving plates are rather limited. The earliest research on transverse vibrations of in-plane moving plates can be dated back to 1982, Ulsoy and Mote [1] reduced the large-scale band saw blade to in-plane moving plates and

* Corresponding author at: Shanghai Institute of Applied Mathematics and Mechanics, Shanghai 200072, China.
E-mail address: lqchen@straff.shu.edu.cn (L.-Q. Chen).

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analyzed the coupled transverse and torsional vibrations of band saw blade. Lee and Ng [2] investigated dynamic stability of a moving rectangular plate with four free edges subject to in-plane acceleration and force perturbations based on Hamilton's principle and the assumed mode method. The effects of sinusoidal perturbations in respect of the in-plane acceleration and external loads are then examined using Bolotin's method. Lengoc and Mccallion [3–5] considered cutting conditions on dynamic response of band saw blades. Lin [6,7] investigated the stability of a moving plate with two simply supported and two free edges by using the canonical form of the equations of motion, subjected to a uniform in-plane tension in transport direction. Wang [8] developed a mixed finite element formulation for a moving orthotropic plate based on the Mindlin–Reissner plate model. The formulation leads to an unexpected artificial damping in the presence of a simply supported boundary, which is related to the gyroscopic matrix formulation. Luo and Hutton [9] presented the formulation of a moving triangular isotropic plate element subjected to in-plane forces and gyroscopic forces and compared the results with the Rayleigh–Ritz method. Kim et al. [10] formulated the modal spectral element for thin plates moving with constant speed under a uniform in-plane axial tension. Hatami et al. [11–13] studied free vibration of in-plane moving symmetrically laminated plates subjected to in-plan forces by classical plate theory. Zhou and Wang [14–16] investigated the dynamic characteristics and stability of in-plane moving viscoelastic rectangular thin plates. Hatami et al. [17] developed an exact finite strip method for the free vibration analysis of in-plane moving viscoelastic plates. Banichuk et al. [18] performed a general dynamic analysis and shown that the onset of instability takes place in the form of divergence.

Historically, the nonlinear vibration of in-plane moving plates has been an important subject of research. However, there are just a few publications on the nonlinear vibration of in-plane moving plates. Luo and Hamidzadeh [19] obtained the frequencies and responses of the free vibration of traveling, nonlinear, elastic plates that are based on a nonlinear plate theory. Hatami et al. [20] developed a nonlinear finite element formulation for analysis of axially moving two-dimensional materials, based on the classical thin plate theory. Luo [21] obtained the analytical conditions for chaotic motions of in-plane traveling, thin plates from the incremental energy approach and found that chaotic motion might occur in the small-amplitude oscillations once the geometrical nonlinearity was considered. Luo and Hamidzadeh [22] obtained analytically equilibrium, membrane force and buckling stability of full simply supported nonlinear traveling plates.

Internal resonance is a typical nonlinear phenomenon. However, the investigations on the internal resonance for axially moving structures are very limited even for one-dimensional cases. Chen et al. [23] investigated the nonlinear vibration analysis of axially moving systems based on the multidimensional Lindstedt–Poincaré method and studied the forced response of an axially moving beam with internal resonance between the first two transverse modes. Sze et al. [24] investigated the nonlinear vibration analysis of axially moving beams based on the incremental harmonic balance method and studied the forced response of an axially moving strip with internal resonance between the first two transverse modes. Suweken and Horssen [25] investigated on the transversal vibrations of a conveyor belt with a low and time-varying velocity. For special values of the belt parameters these sumtype and difference type of internal resonances coincide giving rise to even more complicated dynamical behavior. Hang and Chen [26] investigated the combination response analysis of an axially moving beam with internal resonance and obtained the complex internal resonance curves. Hang and Chen [27] investigated nonlinear vibration of axially moving beams with varying velocities based on the incremental harmonic balance method and discussed the critical velocity with internal resonance. Pakdemirli and Özkaya [28] considered three-to-one internal resonance case of a general continuous system with an arbitrary cubic nonlinearity. Özhan and Pakdemirli [29] further extended their treatments on internal resonance to the system with external excitations.

There have been also many publications on the internal resonance for plates and beams. Chang et al. [30] investigated nonlinear vibrations and chaos in harmonically excited rectangular plates with one-to-one internal resonance. Lewandowski [31] studied beams membranes and plates in free vibration backbone curves in cases of internal resonance. Abe et al. [32] studied two-mode responses of thin rectangular laminated plates subjected to a harmonic excitation with internal resonance by using the method of multiple scales. Ribeiro and Petyt [33] investigated nonlinear free vibration of isotropic plates with internal resonance. Leung and Fung [34] introduced a finite element method based on the virtual work principle to determine the steady state response of frames in free or forced periodic vibration with internal resonance. There have been no works on in-plane moving plates with internal resonances. To address the lacks of research in the aspect, the present investigation focuses on nonlinear free transverse vibrations of in-plane moving plates with internal resonances.

The present paper is organized as follows. Section 2 derives the governing equation and the associated boundary conditions from the Hamilton principle and the Hooke law. Section 3 employs the method of multiple scales to analyze the governing equation under the associated boundary conditions and uses the complex mode approach to analyze and numerically calculate the exact natural frequencies and modes of the linear generating system. Section 4 establishes the solvability conditions without internal resonances and numerically calculates the nonlinear frequencies of the in-plane moving plate. Sections 5 and 6, respectively, investigate 3:1 and 1:1 internal resonances and establish the corresponding solvability conditions. Some numerical examples are presented to demonstrate the variations of the amplitude ratios with the differences of the two frequencies and illustrate the variations at different nonlinear coefficients. Section 7 develops the differential quadrature schemes to confirm the analytical results. Section 8 ends the paper with concluding remarks.

2. Problem formulation

Consider a rectangular thin plate with constant in-plane moving speed Γ in the x direction. The plate has length a, width b, and thickness h in the x, y, and z directions, respectively. Although the transverse vibration is generally coupled with the in-plane vibration, many researchers considered only the transverse vibration in order to derive a tractable and simple equation. It must be remarked that: (1) the equations of motion with in-plane vibrations are also tractable; (2) by neglecting these vibrations, one obtains a too stiff model. In fact, the transverse vibration is more vulnerable than the in-plane vibration.

The total kinetic energy T of structure mass for the in-plane moving plate under in-plane stresses can be written as

$$T=\int_0^a\int_0^b\frac{\rho}{2}\left[\Gamma^2+\left(\frac{\mathrm{d}w}{\mathrm{d}t}\right)^2\right]\mathrm{d}y\,\mathrm{d}x=\int_0^a\int_0^b\frac{\rho}{2}\left[\Gamma^2+(w_{,t}+\Gamma w_{,x})^2\right]\mathrm{d}y\,\mathrm{d}x \tag{1}$$

where a comma preceding x or t denotes partial differentiation with respect to x or t, ρ is the plate's mass per unit of area, and w is the total transverse displacement. The first term in Eq. (1) represents the kinetic energy associated with the longitudinal in-plane motion of the plate, and the second term represents the kinetic energy associated with the transverse vibration.

The total potential energy U for the in-plane moving plate under in-plane stresses includes the strain energy U_b due to bending and the energy U_g due to the effect of an in-plane force per unit width, N_{x0}, on the transverse deflection

$$U=U_b+U_g \tag{2}$$

The strain energy U_b can be written as

$$U_b=\int_0^a\int_0^b\int_{-h/2}^{h/2}(\sigma_x\varepsilon_x+\sigma_y\varepsilon_y+\tau_{xy}\gamma_{xy})\,\mathrm{d}z\,\mathrm{d}y\,\mathrm{d}x \tag{3}$$

where σ_x and σ_y are the normal stress components, τ_{xy} is the shear stress components, ε_x and ε_y are the normal strain components, and γ_{xy} is the shear strain components.

The strain–displacement relations for the classical thin plate including the nonlinearity due to midline strething can be written as

$$\varepsilon_x=-zw_{,xx}+\frac{1}{2}w_{,x}^2,\quad \varepsilon_y=-zw_{,yy}+\frac{1}{2}w_{,y}^2,\quad \gamma_{xy}=-2zw_{,xy}+w_{,x}w_{,y} \tag{4}$$

The material of the plate is linear elastic, defined by Hooke's law

$$\sigma_x=\frac{E}{1-\mu^2}(\varepsilon_x+\mu\varepsilon_y),\quad \sigma_y=\frac{E}{1-\mu^2}(\varepsilon_y+\mu\varepsilon_x),\quad \tau_{xy}=\frac{E}{2(1+\mu)}\gamma_{xy} \tag{5}$$

where E is Young's modulus and μ is Poisson's ratio.

The energy U_g due to the in-plane force per unit width can be written as

$$U_g=\int_0^a\int_0^b\frac{1}{2}N_{x0}w_{,x}^2\,\mathrm{d}y\,\mathrm{d}x \tag{6}$$

The Hamilton principle takes the familiar form:

$$\delta\int_{t_1}^{t_2}(T-U)\,\mathrm{d}t=0 \tag{7}$$

Performing the variation on the energy terms and substituting the results into Eq. (7) give the governing partial differential equation of the transverse nonlinear vibration for the in-plane moving plate in free vibration

$$\begin{aligned}&\rho(w_{,tt}+2\Gamma w_{,xt}+\Gamma^2w_{,xx})+D(w_{,xxxx}+2w_{,xxyy}+w_{,yyyy})-N_{x0}w_{,xx}\\&=6\frac{D}{h^2}\left[(w_{,x}^2w_{,xx}+2w_{,x}w_{,y}w_{,xy}+w_{,y}^2w_{,yy})+\mu(w_{,y}^2w_{,xx}-2w_{,x}w_{,y}w_{,xy}+w_{,x}^2w_{,yy})\right]\end{aligned} \tag{8}$$

and the boundary conditions of the in-plane moving plate with four edges simple supports

$$(w)_{x=0,a}=0,\quad (w_{,xx})_{x=0,a}=0;\quad (w)_{y=0,b}=0,\quad (w_{,yy})_{y=0,b}=0 \tag{9}$$

where $D=Eh^3/[12(1-\mu^2)]$.

It is convenient to introduce dimensionless parameters and variables

$$w\leftrightarrow\frac{w}{\sqrt{k\varepsilon}}h,\quad t\leftrightarrow\frac{t}{a}\sqrt{\frac{N_{x0}}{\rho}},\quad x\leftrightarrow\frac{x}{a},\quad y\leftrightarrow\frac{y}{b},\quad \zeta=\frac{D}{N_{x0}a^2},\quad \gamma=\Gamma\sqrt{\frac{\rho}{N_{x0}}},\quad \xi=\frac{a}{b} \tag{10}$$

Substituting Eq. (10) into Eqs. (8) and (9), one obtains the dimensionless governing partial differential equation

$$\begin{aligned}&w_{,tt}+2\gamma w_{,xt}+(\gamma^2-1)w_{,xx}+\zeta(w_{,xxxx}+2\xi^2w_{,xxyy}+\xi^4w_{,yyyy})\\&=6\varepsilon k\zeta[(w_{,xx}+\xi^2\mu w_{,yy})w_{,x}^2+\xi^2(\mu w_{,xx}+\xi^2w_{,yy})w_{,y}^2+2\xi^2(1-\mu)w_{,x}w_{,y}w_{,xy}]\end{aligned} \tag{11}$$

and the dimensionless boundary conditions

$$(w)_{x=0,1}=0, \quad (w_{,xx})_{x=0,1}=0; \quad (w)_{y=0,1}=0, \quad (w_{,yy})_{y=0,1}=0 \tag{12}$$

3. Multi-scale analysis and solution of the linear generating system

The solutions to Eq. (11) can be assumed as

$$w(x,y,t;\varepsilon)=w_0(x,y,T_0,T_1)+\varepsilon w_1(x,y,T_0,T_1)+O(\varepsilon^2) \tag{13}$$

where $T_0=t$ and $T_1=\varepsilon t$ are, respectively, the fast and slow time scales in the method of multiple scales.

Substitution of Eq. (13) and the following relationship:

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial T_0}+\varepsilon\frac{\partial}{\partial T_1}+O(\varepsilon^2), \quad \frac{\mathrm{d}^2}{\mathrm{d}t^2}=\frac{\partial^2}{\partial T_0^2}+2\varepsilon\frac{\partial^2}{\partial T_0\partial T_1}+O(\varepsilon^2) \tag{14}$$

into Eqs. (11) and (12) and then equalization of coefficients of ε^0 and ε^1 in the resulting equations lead to:

$$\varepsilon^0: w_{0,T_0T_0}+2\gamma w_{0,xT_0}+(\gamma^2-1)w_{0,xx}+\zeta(w_{0,xxxx}+2\xi^2 w_{0,xxyy}+\xi^4 w_{0,yyyy})=0 \tag{15}$$

$$\varepsilon^0: (w_0)_{x=0,1}=0, \quad (w_{0,xx})_{x=0,1}=0; \quad (w_0)_{y=0,1}=0, \quad (w_{0,yy})_{y=0,1}=0 \tag{16}$$

$$\begin{aligned}\varepsilon^1: w_{1,T_0T_0}&+2\gamma w_{1,xT_0}+(\gamma^2-1)w_{1,xx}+\zeta(w_{1,xxxx}+2\xi^2 w_{1,xxyy}+\xi^4 w_{1,yyyy})\\&=6k\zeta[(w_{0,xx}+\xi^2\mu w_{0,yy})w_{0,x}^2+\xi^2(\mu w_{0,xx}+\xi^2 w_{0,yy})w_{0,y}^2\\&\quad+2\xi^2(1-\mu)w_{0,x}w_{0,y}w_{0,xy}]-2(w_{0,T_0T_1}+\gamma w_{0,xT_1})\end{aligned} \tag{17}$$

$$\varepsilon^1: (w_1)_{x=0,1}=0, \quad (w_{1,xx})_{x=0,1}=0; \quad (w_1)_{y=0,1}=0, \quad (w_{1,yy})_{y=0,1}=0 \tag{18}$$

The solution to Eq. (15) can be assumed as

$$w_0(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\psi_{mn}(x,y)A_{mn}(T_1)\,\mathrm{e}^{i\omega_{mn}T_0}+cc \tag{19}$$

where A_{mn} denotes a complex function to be determined later, $\psi_{mn}(x, y)$ is the mnth mode function of the system, ω_{mn} is the mnth natural frequency of the system, and cc stands for complex conjugate of the proceeding terms.

Substitution of Eq. (19) into Eq. (15) leads to

$$-\omega_{mn}^2\psi_{mn}+2i\omega_{mn}\gamma\psi_{mn,x}+(\gamma^2-1)\psi_{mn,xx}+\zeta(\psi_{mn,xxxx}+2\xi^2\psi_{mn,xxyy}+\xi^4\psi_{mn,yyyy})=0 \tag{20}$$

The solution to Eq. (20) can be assumed as

$$\psi_{mn}(x,y)=\phi_n(x)\varphi_m(y) \tag{21}$$

Substitution of Eq. (21) into the last two terms of Eq. (16) leads to

$$\varphi_m(y)=\sin(m\pi y) \tag{22}$$

Substitution of Eq. (22) into Eq. (20), multiplying $\varphi_m(y)$, and integrating from y=0 to 1, then applying the orthogonal condition, Eq. (20) yields

$$\zeta\phi_{n,xxxx}+(\gamma^2+2B_2\zeta\xi^2-1)\phi_{n,xx}+2i\omega_{mn}\gamma\phi_{n,x}+(B_4\zeta\xi^4-\omega_{mn}^2)\phi_n=0 \tag{23}$$

where

$$B_2(m)=\int_0^1\varphi_{m,yy}\varphi_m\,\mathrm{d}y\Big/\int_0^1\varphi_m^2\,\mathrm{d}y, \quad B_4(m)=\int_0^1\varphi_{m,yyyy}\varphi_m\,\mathrm{d}y\Big/\int_0^1\varphi_m^2\,\mathrm{d}y \tag{24}$$

The solution to ordinary differential equation (23) can be expressed by

$$\phi_n(x)=C_{1n}(\mathrm{e}^{i\beta_{1n}x}+C_{2n}\,\mathrm{e}^{i\beta_{2n}x}+C_{3n}\,\mathrm{e}^{i\beta_{3n}x}+C_{4n}\,\mathrm{e}^{i\beta_{4n}x}) \tag{25}$$

Substitution of Eq. (25) into Eq. (23) leads to

$$\zeta\beta_{jn}^4-(\gamma^2+2B_2\zeta\xi^2-1)\beta_{jn}^2-2\omega_{mn}\gamma\beta_{jn}+(B_4\zeta\xi^4-\omega_{mn}^2)=0 \quad (j=1,2,3,4) \tag{26}$$

When the boundary conditions of the in-plane moving plate are simple supports on x=0 and 1, the modal function in the x direction is [35]

$$\begin{aligned}\phi_n(x)=c_1\Bigg\{&e^{i\beta_{1n}x}-\frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{3n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(e^{i\beta_{3n}}-e^{i\beta_{2n}})}e^{i\beta_{2n}x}-\frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{2n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(e^{i\beta_{2n}}-e^{i\beta_{3n}})}e^{i\beta_{3n}x}\\&-\left(1-\frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{3n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{2n}^2)(e^{i\beta_{3n}}-e^{i\beta_{2n}})}+\frac{(\beta_{4n}^2-\beta_{1n}^2)(e^{i\beta_{2n}}-e^{i\beta_{1n}})}{(\beta_{4n}^2-\beta_{3n}^2)(e^{i\beta_{2n}}-e^{i\beta_{3n}})}\right)e^{i\beta_{4n}x}\Bigg\}\end{aligned} \tag{27}$$

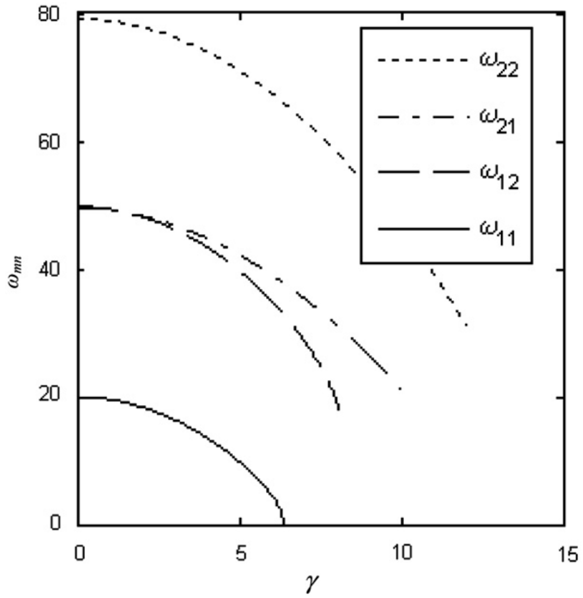

Fig. 1. Dimensionless natural frequencies vs. dimensionless in-plane moving speeds for the in-plane moving plate.

in which β_{jn} (j=1, 2, 3, 4) and the mnth natural frequency ω_{mn} can be solved [36] from Eq. (26) and

$$\left[e^{i(\beta_{1n}+\beta_{2n})}+e^{i(\beta_{3n}+\beta_{4n})}\right](\beta_{1n}^2-\beta_{2n}^2)(\beta_{3n}^2-\beta_{4n}^2)+[e^{i(\beta_{1n}+\beta_{3n})}+e^{i(\beta_{2n}+\beta_{4n})}]$$
$$\times(\beta_{3n}^2-\beta_{1n}^2)(\beta_{2n}^2-\beta_{4n}^2)+[e^{i(\beta_{2n}+\beta_{3n})}+e^{i(\beta_{1n}+\beta_{4n})}](\beta_{2n}^2-\beta_{3n}^2)(\beta_{1n}^2-\beta_{4n}^2)=0 \quad (28)$$

In this paper, numerical calculations are performed to investigate the nonlinear free transverse vibration of an in-plane moving plate with μ=0.3, ζ=1.0, and ξ=1.0. The natural frequencies of the linear generating system can be calculated numerically. Fig. 1 presents the variation of the first four-order dimensionless natural frequencies of the plate with dimensionless in-plane moving speeds. The solid line denotes the first natural frequency ω_{11}, the dashed line denotes the second natural frequency ω_{12}, the dash-dot line denotes the third natural frequency ω_{21}, and the dotted line denotes the fourth natural frequency ω_{22}. They have found that the natural frequencies decrease with increasing in-plane moving speeds. The exact values at which the natural frequencies vanish are called the critical speeds and afterwards the system is unstable about the zero equilibrium. The linear critical in-plane moving speeds for the first two modes are, respectively, γ_{1cr}=6.36227 and γ_{2cr}=7.91739.

To investigate nonlinear free transverse vibration of the in-plane moving plate, the solution to Eq. (15) is assumed to be expressed by

$$w_0(x,y,T_0,T_1)=\psi_{sl}(x,y)A_{sl}(T_1)e^{i\omega_{sl}T_0}+\psi_{s'l'}(x,y)A_{s'l'}(T_1)e^{i\omega_{s'l'}T_0}+cc \quad (29)$$

where ω_{sl} and $\omega_{s'l'}$ are, respectively, the slth and the $s'l'$th natural frequencies of the ε^0-order system defined by Eqs. (26) and (28).

4. No internal resonances

Substitution of Eq. (29) into Eq. (17) yields

$$w_{1,T_0T_0}+2\gamma w_{1,xT_0}+(\gamma^2-1)w_{1,xx}+\zeta(w_{1,xxxx}+2\xi^2 w_{1,xxyy}+\xi^4 w_{1,yyyy})$$
$$=-[E_1A_{sl,T_1}+k(G_{11}A_{sl}^2\overline{A}_{sl}+G_{12}A_{sl}A_{s'l'}\overline{A}_{s'l'})]e^{i\omega_{sl}T_0}$$
$$-[E_2A_{s'l',T_1}+k(G_{22}A_{s'l'}^2\overline{A}_{s'l'}+G_{21}A_{s'l'}A_{sl}\overline{A}_{sl})]e^{i\omega_{s'l'}T_0}+cc+NST \quad (30)$$

where NST stands for non-secular terms and

$$E_j=2(i\omega_h\psi_h+\gamma\psi_{h,x}) \quad (\text{if } j=1,\ h=sl;\ \text{if } j=2,\ h=s'l') \quad (31a)$$

$$G_j=-6\zeta\Big\{\psi_{h,x}^2(\mu\xi^2\overline{\psi}_{h,yy}+\overline{\psi}_{h,xx})+\xi^2\psi_{h,y}^2(\xi^2\overline{\psi}_{h,yy}+\mu\overline{\psi}_{h,xx})$$
$$+2\Big[\psi_{h,x}\overline{\psi}_{h,x}(\mu\xi^2\psi_{h,yy}+\psi_{h,xx})+\xi^2\psi_{h,y}\overline{\psi}_{h,y}(\xi^2\psi_{h,yy}+\mu\psi_{h,xx})$$
$$+(1-\mu)\xi^2(\psi_{h,x}\overline{\psi}_{h,y}\psi_{h,xy}+\overline{\psi}_{h,x}\psi_{h,y}\psi_{h,xy}+\psi_{h,x}\psi_{h,y}\overline{\psi}_{h,xy})\Big]\Big\}$$
$$\times(\text{if } j=11,\ h=sl;\ \text{if } j=22,\ h=s'l') \quad (31b)$$

$$\begin{aligned} G_h = -12\zeta[(1-\mu)\xi^2(&\psi_{i,x}\overline{\psi}_{j,y}\psi_{j,xy}+\psi_{i,x}\psi_{j,y}\overline{\psi}_{j,xy}+\overline{\psi}_{j,x}\psi_{i,y}\psi_{j,xy} \\ &+\psi_{j,x}\psi_{i,y}\overline{\psi}_{j,xy}+\psi_{j,x}\overline{\psi}_{j,y}\psi_{i,xy}+\overline{\psi}_{j,x}\psi_{j,y}\psi_{i,xy}) \\ &+\xi^2\psi_{i,y}\overline{\psi}_{j,y}(\xi^2\psi_{j,yy}+\mu\psi_{j,xx})+\xi^2\psi_{j,y}\overline{\psi}_{j,y}(\xi^2\psi_{i,yy}+\mu\psi_{i,xx}) \\ &+\xi^2\psi_{i,y}\psi_{j,y}(\xi^2\overline{\psi}_{j,yy}+\mu\overline{\psi}_{j,xx})+\psi_{i,x}\psi_{j,x}(\mu\xi^2\overline{\psi}_{j,yy}+\overline{\psi}_{j,xx}) \\ &+\psi_{i,x}\overline{\psi}_{j,x}(\mu\xi^2\psi_{j,yy}+\psi_{j,xx})+\psi_{j,x}\overline{\psi}_{j,x}(\mu\xi^2\psi_{i,yy}+\psi_{i,xx})] \\ &\times(\text{if}\;\; h=12,\; i=sl,\; j=s'l';\;\; h=21,\; i=s'l',\; j=sl) \end{aligned} \tag{31c}$$

It can be checked that the linear part of the mass and stiffness operators in governing equation (15) is symmetric and the gyroscopic operator is skew symmetric under the corresponding boundary conditions (16). The solvability condition presented by Chen and Zu [37] demands the orthogonal relationships

$$\langle E_1A_{sl},_{T_1}+k(G_{11}A_{sl}^2\overline{A}_{sl}+G_{12}A_{sl}A_{s'l'}\overline{A}_{s'l'}),\;\psi_{sl}\rangle=0 \tag{32a}$$

$$\langle E_2A_{s'l'},_{T_1}+k(G_{22}A_{s'l'}^2\overline{A}_{s'l'}+G_{21}A_{s'l'}A_{sl}\overline{A}_{sl}),\;\psi_{s'l'}\rangle=0 \tag{32b}$$

where the inner product is defined for complex functions f and g on [0,1] as

$$\langle f,g\rangle=\int_0^1 f\overline{g}\,\mathrm{d}x \tag{33}$$

Application of the distributive law of the inner product to Eqs. (32a) and (32b) leads to

$$A_{sl},_{T_1}+k(g_{11}A_{sl}^2\overline{A}_{sl}+g_{12}A_{sl}A_{s'l'}\overline{A}_{s'l'})=0 \tag{34a}$$

$$A_{s'l'},_{T_1}+k(g_{22}A_{s'l'}^2\overline{A}_{s'l'}+g_{21}A_{s'l'}A_{sl}\overline{A}_{sl})=0 \tag{34b}$$

where

$$\left.\begin{aligned} g_{11}&=\int_0^1\int_0^1 G_{11}\overline{\psi}_{sl}\,\mathrm{d}x\,\mathrm{d}y\Big/\int_0^1\int_0^1 E_1\overline{\psi}_{sl}\,\mathrm{d}x\,\mathrm{d}y, \quad g_{22}=\int_0^1\int_0^1 G_{22}\overline{\psi}_{s'l'}\,\mathrm{d}x\,\mathrm{d}y\Big/\int_0^1\int_0^1 E_2\overline{\psi}_{s'l'}\,\mathrm{d}x\,\mathrm{d}y, \\ g_{12}&=\int_0^1\int_0^1 G_{12}\overline{\psi}_{sl}\,\mathrm{d}x\,\mathrm{d}y\Big/\int_0^1\int_0^1 E_1\overline{\psi}_{sl}\,\mathrm{d}x\,\mathrm{d}y, \quad g_{21}=\int_0^1\int_0^1 G_{21}\overline{\psi}_{s'l'}\,\mathrm{d}x\,\mathrm{d}y\Big/\int_0^1\int_0^1 E_2\overline{\psi}_{s'l'}\,\mathrm{d}x\,\mathrm{d}y \end{aligned}\right\} \tag{35}$$

It can be numerically demonstrated that g_{11}, g_{12}, g_{21}, and g_{22} are negative imaginary numbers.

Express the solution to Eqs. (34a) and (34b) in polar form

$$A_{sl}=\alpha_{sl}(T_1)e^{i\beta_{sl}(T_1)}, \quad A_{s'l'}=\alpha_{s'l'}(T_1)e^{i\beta_{s'l'}(T_1)} \tag{36}$$

Substituting Eq. (36) into Eq. (34) yields

$$\left.\begin{aligned} &\alpha_{sl},_{T_1}=0 \\ &\alpha_{sl}\beta_{sl},_{T_1}=-kg_{11}^I\alpha_{sl}^3-kg_{12}^I\alpha_{s'l'}^2\alpha_{sl} \\ &\alpha_{s'l'},_{T_1}=0 \\ &\alpha_{s'l'}\beta_{s'l'},_{T_1}=-kg_{22}^I\alpha_{s'l'}^3-kg_{21}^I\alpha_{sl}^2\alpha_{s'l'} \end{aligned}\right\} \tag{37}$$

where g_i^I (i=11, 12, 21, 22) is imaginary part of the g_i (i=11, 12, 21, 22). Eq. (37) obviously has a zero solution. If we assume that there is a non-zero solution, there are two possibilities: (1) $\alpha_{s'l'}=0$ and $\alpha_{sl}\neq0$; (2) $\alpha_{sl}=0$ and $\alpha_{s'l'}\neq0$; and (3) $\alpha_{sl}\neq0$ and $\alpha_{s'l'}\neq0$.

In the first uncoupled case, substituting $\alpha_{s'l'}=0$ and $\alpha_{sl}\neq0$ into Eq. (37) and integrating the results yield

$$\alpha_{sl}=\alpha_{sl0}, \quad \beta_{sl}=-kg_{11}^I\alpha_{sl0}^2T_1+\beta_{sl0} \tag{38}$$

where the initial amplitude α_{sl0} and the phase β_{sl0} are constants. Inserting Eq. (38) into Eq. (36) and then inserting the resulting equation into Eq. (29) gives the slth nonlinear frequency:

$$\Omega_{sl}=\omega_{sl}-\varepsilon kg_{11}^I\alpha_{sl0}^2 \tag{39}$$

In the second uncoupled case, we can similarly obtain the $s'l'$th nonlinear frequency:

$$\Omega_{s'l'}=\omega_{s'l'}-\varepsilon kg_{22}^I\alpha_{s'l'0}^2 \tag{40}$$

where the initial amplitude α_{sl0} and the phase β_{sl0} are constants.

In the third coupled case, substituting $\alpha_{sl}\neq0$ and $\alpha_{s'l'}\neq0$ into Eq. (37) yields

$$\frac{\alpha_{sl0}^2}{\alpha_{s'l'0}^2}=-\frac{g_{12}^I}{g_{11}^I}, \quad \frac{\alpha_{sl0}^2}{\alpha_{s'l'0}^2}=-\frac{g_{22}^I}{g_{21}^I} \tag{41}$$

The two equations in Eq. (41) cannot come into existence at the same time. Thus, the assumption is wrong.

It can be found that the *sl*th mode and the *s′l′*th mode cannot be non-zero at the same time from the three cases. Therefore, it is feasible to investigate resonances with the possible contributions of modes not involved in the resonance.

Based on Eqs. (39) and (40), the nonlinear frequencies of nonlinear free vibration of the in-plane moving plate without internal resonances can be numerically calculated. Because ε is a characterization parameter, there is no specific physical meaning. In this paper, let $\varepsilon=1$. Fig. 2 shows the relationship of the in-plane moving plate for $k=1$ between the nonlinear frequencies and the initial amplitudes at different in-plane moving speeds for $\varepsilon=1$, respectively. Both modes predict the same trends of the nonlinear free vibration frequencies varying with the initial amplitudes and the in-plane moving speeds. The dotted lines denote $\gamma=2.8$, the dashed lines denote $\gamma=2.8$, and the solid lines denote $\gamma=3.0$. They have found that the nonlinear frequencies increase with increase in initial amplitudes. When the initial amplitudes and the phases are zero, Eqs. (39) and (40) yield the frequencies of the corresponding linear system. The larger in-plane moving speed results in the smaller frequencies and the rapid increase of the frequencies with the initial amplitudes.

Fig. 3(a) and (b) show the relationship of the in-plane moving plate for $\varepsilon=1$ between the nonlinear frequencies and the initial amplitudes at different nonlinear coefficients for $\gamma=3.0$, respectively. The dotted lines denote $k=0.6$, the dashed lines denote $k=0.8$, and the solid lines denote $k=1.0$ in Fig. 3. Both modes predict the same trends of the nonlinear free vibration frequencies varying with the nonlinear coefficients. The larger nonlinear coefficient results in the rapid increase of the nonlinear frequencies with the initial amplitudes, and the increase becomes substantial when the in-plane moving speed is close to the critical one. Besides, the effects are more significant in the higher order mode. The nonlinear frequencies involving higher order modes are more sensitive to the initial amplitudes, the in-plane moving speeds, and the nonlinear coefficients.

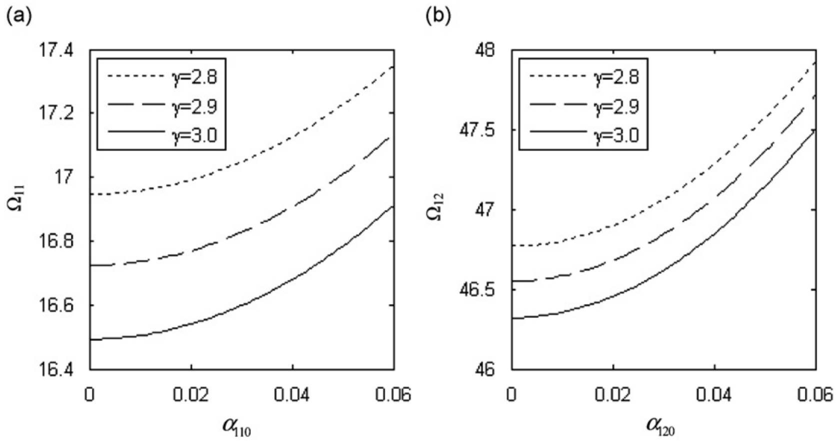

Fig. 2. The relationship between the nonlinear frequencies and the initial amplitudes at different in-plane moving speed: (a) the first mode and (b) the second mode.

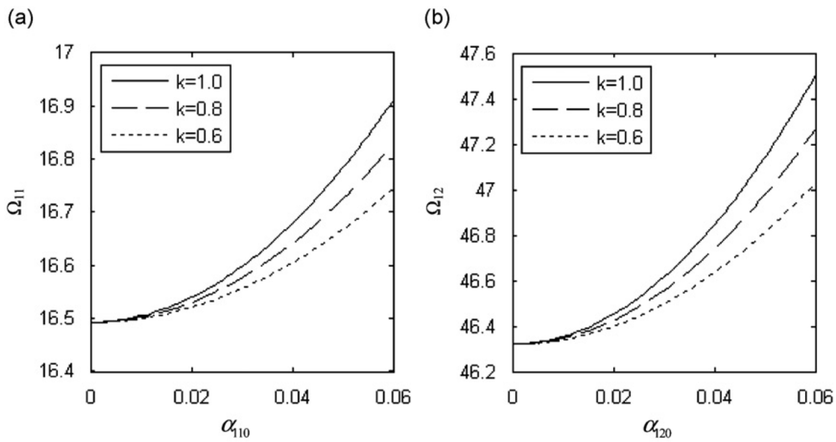

Fig. 3. The relationship between the nonlinear frequencies and the initial amplitudes at different nonlinear coefficients: (a) the first mode and (b) the second mode.

5. 3:1 internal resonance

If any two natural frequencies of generating autonomous linear system (15) are approximately in the ratio of 3:1, internal resonance may occur. A detuning parameter σ is introduced to quantify the deviation of $\omega_{s'l'}$ from $3\omega_{sl}$, and $\omega_{s'l'}$ is described by

$$\omega_{s'l'} = 3\omega_{sl} + \varepsilon\sigma \tag{42}$$

Substitution of Eqs. (29) and (42) into Eq. (17) yields

$$\begin{aligned} &w_{1,T_0T_0} + 2\gamma w_{1,xT_0} + (\gamma^2-1)w_{1,xx} + \zeta(w_{1,xxxx} + 2\xi^2 w_{1,xxyy} + \xi^4 w_{1,yyyy}) \\ &\quad = -[E_1 A_{sl,T_1} + k(G_{11}A_{sl}^2\overline{A}_{sl} + G_{12}A_{sl}A_{s'l'}\overline{A}_{s'l'} + H_1 A_{s'l'}\overline{A}_{sl}^2 e^{i\sigma T_1})]e^{i\omega_{sl}T_0} \\ &\quad -[E_2 A_{s'l',T_1} + k(G_{22}A_{s'l'}^2\overline{A}_{s'l'} + G_{21}A_{s'l'}A_{sl}\overline{A}_{sl} + H_2 A_{sl}^3 e^{-i\sigma T_1})]e^{i\omega_{s'l'}T_0} + cc + NST \end{aligned} \tag{43}$$

where *NST* stands for non-secular terms and

$$E_j = 2(i\omega_h\psi_h + \gamma\psi_{h,x}) \quad (\text{if } j=1,\ h=sl;\ \text{if } j=2,\ h=s'l') \tag{44a}$$

$$\begin{aligned} G_j = &-6\zeta\{\psi_{h,x}^2(\mu\xi^2\overline{\psi}_{h,yy} + \overline{\psi}_{h,xx}) + \xi^2\psi_{h,y}^2(\xi^2\overline{\psi}_{h,yy} + \mu\overline{\psi}_{h,xx}) \\ &+ 2[\psi_{h,x}\overline{\psi}_{h,x}(\mu\xi^2\psi_{h,yy} + \psi_{h,xx}) + \xi^2\psi_{h,y}\overline{\psi}_{h,y}(\xi^2\psi_{h,yy} + \mu\psi_{h,xx}) \\ &+ (1-\mu)\xi^2(\psi_{h,x}\overline{\psi}_{h,y}\psi_{h,xy} + \overline{\psi}_{h,x}\psi_{h,y}\psi_{h,xy} + \psi_{h,x}\psi_{h,y}\overline{\psi}_{h,xy})]\} \\ &(\text{if } j=11,\ h=sl;\ \text{if } j=22,\ h=s'l') \end{aligned} \tag{44b}$$

$$\begin{aligned} G_h = &-12\zeta[(1-\mu)\xi^2(\psi_{i,x}\overline{\psi}_{j,y}\psi_{j,xy} + \psi_{i,x}\psi_{j,y}\overline{\psi}_{j,xy} + \overline{\psi}_{j,x}\psi_{i,y}\psi_{j,xy} \\ &+ \psi_{j,x}\psi_{i,y}\overline{\psi}_{j,xy} + \psi_{j,x}\overline{\psi}_{j,y}\psi_{i,xy} + \overline{\psi}_{j,x}\psi_{j,y}\psi_{i,xy}) \\ &+ \xi^2\psi_{i,y}\overline{\psi}_{j,y}(\xi^2\psi_{j,yy} + \mu\psi_{j,xx}) + \xi^2\psi_{j,y}\overline{\psi}_{j,y}(\xi^2\psi_{i,yy} + \mu\psi_{i,xx}) \\ &+ \xi^2\psi_{i,y}\psi_{j,y}(\xi^2\overline{\psi}_{j,yy} + \mu\overline{\psi}_{j,xx}) + \psi_{i,x}\psi_{j,x}(\mu\xi^2\overline{\psi}_{j,yy} + \overline{\psi}_{j,xx}) \\ &+ \psi_{i,x}\overline{\psi}_{j,x}(\mu\xi^2\psi_{j,yy} + \psi_{j,xx}) + \psi_{j,x}\overline{\psi}_{j,x}(\mu\xi^2\psi_{i,yy} + \psi_{i,xx})] \\ &(\text{if } h=12,\ i=sl, j=s'l';\ h=21,\ i=s'l', j=sl) \end{aligned} \tag{44c}$$

$$\begin{aligned} H_1 = &-6\zeta\{\overline{\psi}_{sl,x}^2(\mu\xi^2\psi_{s'l',yy} + \psi_{s'l',xx}) + \xi^2\overline{\psi}_{sl,y}^2(\xi^2\psi_{s'l',yy} + \mu\psi_{s'l',xx}) \\ &+ 2[(1-\mu)\xi^2(\overline{\psi}_{sl,x}\psi_{s'l',y}\overline{\psi}_{sl,xy} + \overline{\psi}_{sl,x}\overline{\psi}_{sl,y}\psi_{s'l',xy} + \psi_{s'l',x}\overline{\psi}_{sl,y}\overline{\psi}_{sl,xy}) \\ &+ \overline{\psi}_{sl,x}\psi_{s'l',x}(\mu\xi^2\overline{\psi}_{sl,yy} + \overline{\psi}_{sl,xx}) + \xi^2\overline{\psi}_{sl,y}\psi_{s'l',y}(\xi^2\overline{\psi}_{sl,yy} + \mu\overline{\psi}_{sl,xx})]\} \end{aligned} \tag{44d}$$

$$\begin{aligned} H_2 = &-6\zeta[\psi_{sl,x}^2(\mu\xi^2\psi_{sl,yy} + \psi_{sl,xx}) + \xi^2\psi_{sl,y}^2(\xi^2\psi_{sl,yy} + \mu\psi_{sl,xx}) \\ &+ 2(1-\mu)\xi^2\psi_{sl,x}\psi_{sl,y}\psi_{sl,xy}] \end{aligned} \tag{44e}$$

The solvability condition demands the orthogonal relationships

$$\left.\begin{aligned} &\left\langle E_1 A_{sl,T_1} + k(G_{11}A_{sl}^2\overline{A}_{sl} + G_{12}A_{sl}A_{s'l'}\overline{A}_{s'l'} + H_1 A_{s'l'}\overline{A}_{sl}^2 e^{i\sigma T_1}),\ \psi_{sl}\right\rangle = 0, \\ &\langle E_2 A_{s'l',T_1} + k(G_{22}A_{s'l'}^2\overline{A}_{s'l'} + G_{21}A_{s'l'}A_{sl}\overline{A}_{sl} + H_2 A_{sl}^3 e^{-i\sigma T_1}),\ \psi_{s'l'}\rangle = 0 \end{aligned}\right\} \tag{45}$$

Application of the distributive law of the inner product to Eq. (45) leads to

$$\left.\begin{aligned} &A_{sl,T_1} + k(g_{11}A_{sl}^2\overline{A}_{sl} + g_{12}A_{sl}A_{s'l'}\overline{A}_{s'l'} + h_1 A_{s'l'}\overline{A}_{sl}^2 e^{i\sigma T_1}) = 0, \\ &A_{s'l',T_1} + k(g_{22}A_{s'l'}^2\overline{A}_{s'l'} + g_{21}A_{s'l'}A_{sl}\overline{A}_{sl} + h_2 A_{sl}^3 e^{-i\sigma T_1}) = 0 \end{aligned}\right\} \tag{46}$$

where

$$\left.\begin{aligned} &g_{11} = \int_0^1\!\!\int_0^1 G_{11}\overline{\psi}_{sl}\,dx\,dy \Big/ \int_0^1\!\!\int_0^1 E_1\overline{\psi}_{sl}\,dx\,dy, \quad g_{22} = \int_0^1\!\!\int_0^1 G_{22}\overline{\psi}_{s'l'}\,dx\,dy \Big/ \int_0^1\!\!\int_0^1 E_2\overline{\psi}_{s'l'}\,dx\,dy, \\ &g_{12} = \int_0^1\!\!\int_0^1 G_{12}\overline{\psi}_{sl}\,dx\,dy \Big/ \int_0^1\!\!\int_0^1 E_1\overline{\psi}_{sl}\,dx\,dy, \quad g_{21} = \int_0^1\!\!\int_0^1 G_{21}\overline{\psi}_{s'l'}\,dx\,dy \Big/ \int_0^1\!\!\int_0^1 E_2\overline{\psi}_{s'l'}\,dx\,dy, \\ &h_1 = \int_0^1\!\!\int_0^1 H_1\overline{\psi}_{sl}\,dx\,dy \Big/ \int_0^1\!\!\int_0^1 E_1\overline{\psi}_{sl}\,dx\,dy, \quad h_2 = \int_0^1\!\!\int_0^1 H_2\overline{\psi}_{s'l'}\,dx\,dy \Big/ \int_0^1\!\!\int_0^1 E_2\overline{\psi}_{s'l'}\,dx\,dy \end{aligned}\right\} \tag{47}$$

It can be numerically demonstrated that g_{11}, g_{12}, g_{21}, and g_{22} are negative imaginary numbers and h_1 and h_2 are a complex numbers.

Express the solution to Eq. (46) in polar form

$$A_{sl} = \alpha_{sl}(T_1)e^{i\beta_{sl}(T_1)}, \quad A_{s'l'} = \alpha_{s'l'}(T_1)e^{i\beta_{s'l'}(T_1)} \tag{48}$$

Substituting Eq. (48) into Eq. (46) yields

$$\alpha_{sl},_{T_1} = -k[h_1^R\cos(\theta) - h_1^I\sin(\theta)]\alpha_{s'l'}\alpha_{sl}^2 \tag{49a}$$

$$\alpha_{sl}\beta_{sl},_{T_1} = -k\alpha_{sl}\{g_{11}^I\alpha_{sl}^2 + g_{12}^I\alpha_{s'l'}^2 + [h_1^I\cos(\theta) + h_1^R\sin(\theta)]\alpha_{sl}\alpha_{s'l'}\} \tag{49b}$$

$$\alpha_{s'l'},_{T_1} = -k[h_2^R\cos(\theta) + h_2^I\sin(\theta)]\alpha_{sl}^3 \tag{49c}$$

$$\alpha_{s'l'}\beta_{s'l'},_{T_1} = -k\left\{g_{22}^I\alpha_{s'l'}^3 + g_{21}^I\alpha_{sl}^2\alpha_{s'l'} + \left[h_2^I\cos(\theta) - h_2^R\sin(\theta)\right]\alpha_{sl}^3\right\} \tag{49d}$$

where g_i^I (i=11, 12, 21, 22) is imaginary part of the g_i (i=11, 12, 21, 22), h_i^R and h_i^I (i=1, 2) are real part and imaginary part of the h_i (i=1, 2) respectively, and $\theta = \beta_{s'l'} - 3\beta_{sl} + \sigma T_1$.

There are two possibilities: (1) α_{sl}=0 and $\alpha_{s'l'} \neq 0$; and (2) $\alpha_{sl} \neq 0$ and $\alpha_{s'l'} \neq 0$.

In the first uncoupled case, we can similarly obtain the *s'l'*th nonlinear frequency:

$$\Omega_{s'l'} = \omega_{s'l'} - \varepsilon k g_{22}^I\alpha_{s'l'0}^2 \tag{50}$$

In the second coupled case, differentiating θ once with respect to T_1 and using Eqs. (49b) and (49d), we obtain

$$\begin{aligned}\alpha_{sl}\alpha_{s'l'}\theta,_{T_1} &= \alpha_{sl}\alpha_{s'l'}\sigma + k(3g_{11}^I - g_{21}^I)\alpha_{s'l'}\alpha_{sl}^3 + k(3g_{12}^I - g_{22}^I)\alpha_{sl}\alpha_{s'l'}^3 \\ &\quad + 3k[h_1^I\cos(\theta) + h_1^R\sin(\theta)]\alpha_{sl}^2\alpha_{s'l'}^2 - k[h_2^I\cos(\theta) - h_2^R\sin(\theta)]\alpha_{sl}^4\end{aligned} \tag{51}$$

For steady-state solutions, the amplitudes α_{sl} and $\alpha_{s'l'}$ and the new phase θ angle should be constant. Then we obtain

$$-k[h_1^R\cos(\theta) - h_1^I\sin(\theta)]\alpha_{s'l'}\alpha_{sl}^2 = 0 \tag{52a}$$

$$-k[h_2^R\cos(\theta) + h_2^I\sin(\theta)]\alpha_{sl}^3 = 0 \tag{52b}$$

$$\begin{aligned}&\alpha_{sl}\alpha_{s'l'}\sigma + k(3g_{11}^I - g_{21}^I)\alpha_{s'l'}\alpha_{sl}^3 + k(3g_{12}^I - g_{22}^I)\alpha_{sl}\alpha_{s'l'}^3 \\ &+ 3k[h_1^I\cos(\theta) + h_1^R\sin(\theta)]\alpha_{sl}^2\alpha_{s'l'}^2 - k[h_2^I\cos(\theta) - h_2^R\sin(\theta)]\alpha_{sl}^4 = 0\end{aligned} \tag{52c}$$

Using Eqs. (52a) and (52b), we obtain

$$\frac{\sin(\theta)}{\cos(\theta)} = \frac{h_1^R}{h_1^I} = -\frac{h_2^R}{h_2^I} = C \tag{53}$$

Substituting Eq. (53) into Eq. (52) and eliminating θ yields

$$\frac{\sigma}{\alpha_{s'l'}^2} = -k\left[(3g_{11}^I - g_{21}^I)\frac{\alpha_{sl}^2}{\alpha_{s'l'}^2} + (3g_{12}^I - g_{22}^I) \pm \sqrt{1+C^2}\left|\frac{\alpha_{sl}}{\alpha_{s'l'}}\left(h_2^I\frac{\alpha_{sl}^2}{\alpha_{s'l'}^2} - 3h_1^I\right)\right|\right] \tag{54}$$

3:1 internal resonances between the first and second modes and between the third and fourth modes of the in-plane moving plates with k=1 may be activated when γ is near γ=3.60443 and 11.5303, respectively. Because they provide the same trends, we focus on the case 3:1 internal resonances between the first and second modes. Fig. 4 shows variations of the amplitude ratios α_{11}/α_{12} with the differences of the two frequencies σ/α_{12}^2 for this case. It can be found that two solutions exist when $\sigma/\alpha_{12}^2 < 891.38$ or $\sigma/\alpha_{12}^2 > 20{,}806$, four solutions exist when σ/α_{12}^2=891.38, σ/α_{12}^2=4898, or

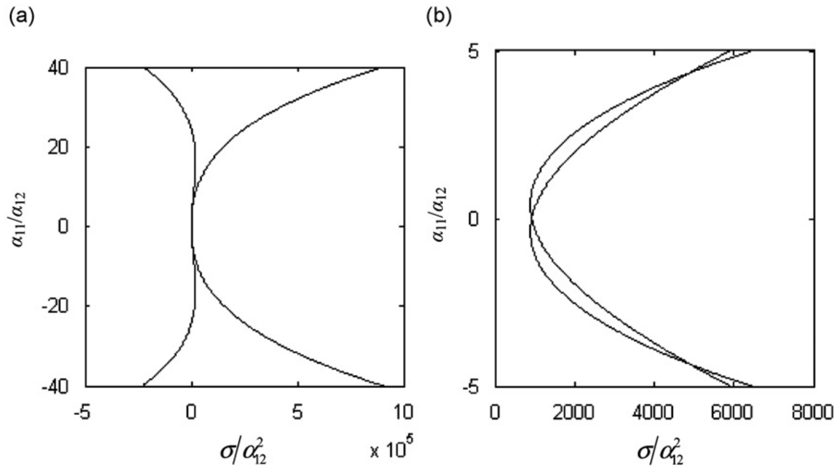

Fig. 4. Variations of the amplitude ratios α_{11}/α_{12} with the differences of the two frequencies σ/α_{21}^2 for the case of 3:1 internal resonance between the first and second modes: (a) overview diagram and (b) detail with enlarged scale.

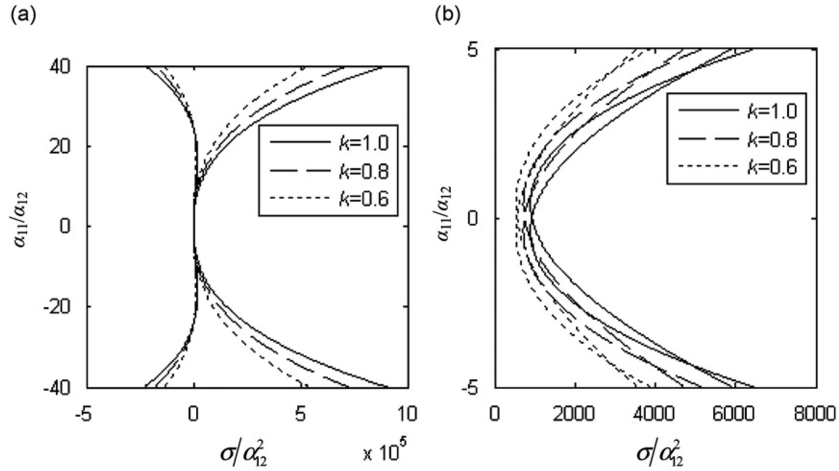

Fig. 5. Variations of the amplitude ratios α_{11}/α_{12} with the differences of the two frequencies σ/α_{21}^2 for the case of 3:1 internal resonance between the first and second modes at different nonlinear coefficient: (a) overview diagram and (b) detail with enlarged scale.

σ/α_{12}^2=20,806, six solutions exist when $891.38<\sigma/\alpha_{12}^2<924.8$, $924.8<\sigma/\alpha_{12}^2<4898$, or $4898<\sigma/\alpha_{12}^2<20{,}806$, and five solutions exist when σ/α_{12}^2=924.8.

Fig. 5 shows variations of the amplitude ratios α_{11}/α_{12} with the differences of the two frequencies σ/α_{12}^2 for the case of 3:1 internal resonance between the first and second modes of the in-plane moving plates at different nonlinear coefficients. The dotted lines denote k=0.6, the dashed lines denote k=0.8, and the solid lines denote k=1.0. The relationship between σ/α_{12}^2 and the nonlinear coefficient at a given α_{11}/α_{12} is linear.

6. 1:1 internal resonance

If any two natural frequencies of linear generating system (15) are approximately in the ratio of 1:1, internal resonance may occur. A detuning parameter σ is introduced to quantify the deviation of ${}_{s'l'}$ from ω_{sl}, and $\omega_{s'l'}$ is described by

$$\omega_{s'l'}=\omega_{sl}+\varepsilon\sigma \tag{55}$$

Substitution of Eqs. (29) and (55) into Eq. (17) yields

$$\begin{aligned}
&w_{1,T_0T_0}+2\gamma w_{1,xT_0}+(\gamma^2-1)w_{1,xx}+\zeta(w_{1,xxxx}+2\xi^2 w_{1,xxyy}+\xi^4 w_{1,yyyy})\\
&=-[E_1A_{sl,T_1}+k(F_1A_{sl}^2\overline{A}_{sl}+G_1A_{sl}A_{s'l'}\overline{A}_{s'l'})]e^{i\omega_{sl}T_0}\\
&-[E_2A_{s'l',T_1}+k(F_2A_{s'l'}^2\overline{A}_{s'l'}+G_2A_{s'l'}A_{sl}\overline{A}_{sl})]e^{i\omega_{s'l'}T_0}\\
&+kH_1A_{s'l'}^2\overline{A}_{sl}e^{2i\omega_{s'l'}T_0-i\omega_{sl}T_0}+kH_2A_{sl}^2\overline{A}_{s'l'}e^{2i\omega_{sl}T_0-i\omega_{s'l'}T_0}+cc+NST
\end{aligned} \tag{56}$$

where NST stands for non-secular terms and

$$E_j=2(i\omega_h\psi_h+\gamma\psi_{h,x})\quad (\text{if } j=1,\ h=sl;\ \text{if } j=2,\ h=s'l') \tag{57a}$$

$$\begin{aligned}
F_j=&-6\zeta\Big\{\psi_{h,x}^2(\mu\xi^2\overline{\psi}_{h,yy}+\overline{\psi}_{h,xx})+\xi^2\psi_{h,y}^2(\xi^2\overline{\psi}_{h,yy}+\mu\overline{\psi}_{h,xx})\\
&+2\Big[\psi_{h,x}\overline{\psi}_{h,x}(\mu\xi^2\psi_{h,yy}+\psi_{h,xx})+\xi^2\psi_{h,y}\overline{\psi}_{h,y}(\xi^2\psi_{h,yy}+\mu\psi_{h,xx})\\
&+(1-\mu)\xi^2(\psi_{h,x}\overline{\psi}_{h,y}\psi_{h,xy}+\overline{\psi}_{h,x}\psi_{h,y}\psi_{h,xy}+\psi_{h,x}\psi_{h,y}\overline{\psi}_{h,xy})]\}\\
&(\text{if } j=1,\ h=sl;\ \text{if } j=2,\ h=s'l')
\end{aligned} \tag{57b}$$

$$\begin{aligned}
G_h=&-12\zeta[(1-\mu)\xi^2(\psi_{i,x}\overline{\psi}_{j,y}\psi_{j,xy}+\psi_{i,x}\psi_{j,y}\overline{\psi}_{j,xy}+\overline{\psi}_{j,x}\psi_{i,y}\psi_{j,xy}\\
&+\psi_{j,x}\psi_{i,y}\overline{\psi}_{j,xy}+\psi_{j,x}\overline{\psi}_{j,y}\psi_{i,xy}+\overline{\psi}_{j,x}\psi_{j,y}\psi_{i,xy})\\
&+\xi^2\psi_{i,y}\overline{\psi}_{j,y}(\xi^2\psi_{j,yy}+\mu\psi_{j,xx})+\xi^2\psi_{j,y}\overline{\psi}_{j,y}(\xi^2\psi_{i,yy}+\mu\psi_{i,xx})\\
&+\xi^2\psi_{i,y}\psi_{j,y}(\xi^2\overline{\psi}_{j,yy}+\mu\overline{\psi}_{j,xx})+\psi_{i,x}\psi_{j,x}(\mu\xi^2\overline{\psi}_{j,yy}+\overline{\psi}_{j,xx})\\
&+\psi_{i,x}\overline{\psi}_{j,x}(\mu\xi^2\psi_{j,yy}+\psi_{j,xx})+\psi_{j,x}\overline{\psi}_{j,x}(\mu\xi^2\psi_{i,yy}+\psi_{i,xx})\Big]\\
&(\text{if } h=1,\ i=sl,\ j=s'l';\ h=2,\ i=s'l',\ j=sl)
\end{aligned} \tag{57c}$$

$$\begin{aligned}H_h=&-6\zeta\Big\{\psi_{i},_x^2(\mu\xi^2\overline{\psi}_{j},_{yy}+\overline{\psi}_{j},_{xx})+\xi^2\psi_{i},_y^2(\xi^2\overline{\psi}_{j},_{yy}+\mu\overline{\psi}_{j},_{xx})\\&+2\Big[(1-\mu)\xi^2(\psi_{i},_x\overline{\psi}_{j},_y\psi_{i},_{xy}+\psi_{i},_x\psi_{i},_y\overline{\psi}_{j},_{xy}+\overline{\psi}_{j},_x\psi_{i},_y\psi_{i},_{xy})\\&+\psi_{i},_x\overline{\psi}_{j},_x(\mu\xi^2\psi_{i},_{yy}+\psi_{i},_{xx})+\xi^2\psi_{i},_y\overline{\psi}_{j},_y(\xi^2\psi_{i},_{yy}+\mu\psi_{i},_{xx})\Big]\Big\}\\&(\text{if } h=1,\, i=s'l',\, j=sl;\; h=2,\, i=sl,\, j=s'l')\end{aligned}\tag{57d}$$

The solvability condition demands the orthogonal relationships

$$\left.\begin{aligned}&\Big\langle E_1A_{sl},_{T_1}+kF_1A_{sl}^2\overline{A}_{sl}+kG_1A_{sl}A_{s'l'}\overline{A}_{s'l'}-kH_1A_{s'l'}^2\overline{A}_{sl}\,e^{2i\sigma T_1}-kH_2A_{sl}^2\overline{A}_{s'l'}\,e^{-i\sigma T_1}+(E_2A_{s'l'},_{T_1}+kF_2A_{s'l'}^2\overline{A}_{s'l'}+kG_2A_{s'l'}A_{sl}\overline{A}_{sl})e^{i\sigma T_1},\ \psi_{sl}\Big\rangle=0,\\&\Big\langle E_2A_{s'l'},_{T_1}+kF_2A_{s'l'}^2\overline{A}_{s'l'}+kG_2A_{s'l'}A_{sl}\overline{A}_{sl}-kH_1A_{s'l'}^2\overline{A}_{sl}\,e^{i\sigma T_1}-kH_2A_{sl}^2\overline{A}_{s'l'}\,e^{-2i\sigma T_1}+(E_1A_{sl},_{T_1}+kF_1A_{sl}^2\overline{A}_{sl}+kG_1A_{sl}A_{s'l'}\overline{A}_{s'l'})e^{-i\sigma T_1},\ \psi_{s'l'}\Big\rangle=0\end{aligned}\right\}\tag{58}$$

Application of the distributive law of the inner product to Eq. (58) leads to

$$\left.\begin{aligned}&e_{11}A_{sl},_{T_1}+kf_{11}A_{sl}^2\overline{A}_{sl}+kg_{11}A_{sl}A_{s'l'}\overline{A}_{s'l'}-kh_{11}A_{s'l'}^2\overline{A}_{sl}\,e^{2i\sigma T_1}-kh_{12}A_{sl}^2\overline{A}_{s'l'}\,e^{-i\sigma T_1}+(e_{12}A_{s'l'},_{T_1}+kf_{12}A_{s'l'}^2\overline{A}_{s'l'}+kg_{12}A_{s'l'}A_{sl}\overline{A}_{sl})e^{i\sigma T_1}=0,\\&e_{22}A_{s'l'},_{T_1}+kf_{22}A_{s'l'}^2\overline{A}_{s'l'}+kg_{22}A_{s'l'}A_{sl}\overline{A}_{sl}-kh_{21}A_{s'l'}^2\overline{A}_{sl}\,e^{i\sigma T_1}-kh_{22}A_{sl}^2\overline{A}_{s'l'}\,e^{-2i\sigma T_1}+(e_{21}A_{sl},_{T_1}+kf_{21}A_{sl}^2\overline{A}_{sl}+kg_{21}A_{sl}A_{s'l'}\overline{A}_{s'l'})e^{-i\sigma T_1}=0\end{aligned}\right\}\tag{59}$$

where

$$\left.\begin{aligned}&e_{11}=\int_0^1\int_0^1E_1\overline{\psi}_{sl}\,\mathrm{d}x\mathrm{d}y,\quad e_{12}=\int_0^1\int_0^1E_2\overline{\psi}_{sl}\,\mathrm{d}x\mathrm{d}y,\quad e_{21}=\int_0^1\int_0^1E_1\overline{\psi}_{s'l'}\,\mathrm{d}x\mathrm{d}y,\quad e_{22}=\int_0^1\int_0^1E_2\overline{\psi}_{s'l'}\,\mathrm{d}x\mathrm{d}y,\\&f_{11}=\int_0^1\int_0^1F_1\overline{\psi}_{sl}\,\mathrm{d}x\mathrm{d}y,\quad f_{12}=\int_0^1\int_0^1F_2\overline{\psi}_{sl}\,\mathrm{d}x\mathrm{d}y,\quad f_{21}=\int_0^1\int_0^1F_1\overline{\psi}_{s'l'}\,\mathrm{d}x\mathrm{d}y,\quad f_{22}=\int_0^1\int_0^1F_2\overline{\psi}_{s'l'}\,\mathrm{d}x\mathrm{d}y,\\&g_{11}=\int_0^1\int_0^1G_1\overline{\psi}_{sl}\,\mathrm{d}x\mathrm{d}y,\quad g_{12}=\int_0^1\int_0^1G_2\overline{\psi}_{sl}\,\mathrm{d}x\mathrm{d}y,\quad g_{21}=\int_0^1\int_0^1G_1\overline{\psi}_{s'l'}\,\mathrm{d}x\mathrm{d}y,\quad g_{22}=\int_0^1\int_0^1G_2\overline{\psi}_{s'l'}\,\mathrm{d}x\mathrm{d}y,\\&h_{11}=\int_0^1\int_0^1H_1\overline{\psi}_{sl}\,\mathrm{d}x\mathrm{d}y,\quad h_{12}=\int_0^1\int_0^1H_2\overline{\psi}_{sl}\,\mathrm{d}x\mathrm{d}y,\quad h_{21}=\int_0^1\int_0^1H_1\overline{\psi}_{s'l'}\,\mathrm{d}x\mathrm{d}y,\quad h_{22}=\int_0^1\int_0^1H_2\overline{\psi}_{s'l'}\,\mathrm{d}x\mathrm{d}y\end{aligned}\right\}\tag{60}$$

It can be numerically demonstrated that $e_{12}=e_{21}=f_{12}=f_{21}=g_{12}=g_{21}=h_{12}=h_{21}=0$.

Eq. (59) can be rewritten as

$$\left.\begin{aligned}&A_{sl},_{T_1}+kS_{11}A_{sl}^2\overline{A}_{sl}+kS_{12}A_{sl}A_{s'l'}\overline{A}_{s'l'}+kR_1A_{s'l'}^2\overline{A}_{sl}\,e^{2i\sigma T_1}=0,\\&A_{s'l'},_{T_1}+kS_{22}A_{s'l'}^2\overline{A}_{s'l'}+kS_{21}A_{s'l'}A_{sl}\overline{A}_{sl}+kR_2A_{sl}^2\overline{A}_{s'l'}\,e^{-2i\sigma T_1}=0\end{aligned}\right\}\tag{61}$$

where

$$S_{11}=f_{11}/e_{11},\quad S_{12}=g_{11}/e_{11},\quad S_{22}=f_{22}/e_{22},\quad S_{21}=g_{22}/e_{22},\quad R_1=-h_{11}/e_{11},\quad R_2=-h_{22}/e_{22}\tag{62}$$

It can be numerically demonstrated that S_{11}, S_{12}, S_{21}, and S_{22} are negative imaginary numbers and R_1 and R_2 are a complex numbers.

Express the solution to Eq. (61) in polar form

$$A_{sl}=\alpha_{sl}(T_1)e^{i\beta_{sl}(T_1)},\quad A_{s'l'}=\alpha_{s'l'}(T_1)e^{i\beta_{s'l'}(T_1)}\tag{63}$$

Substituting Eq. (63) into Eq. (61) yields

$$\alpha_{sl},_{T_1}=-k[R_1^R\cos(2\theta)-R_1^I\sin(2\theta)]\alpha_{sl}\alpha_{s'l'}^2\tag{64a}$$

$$\alpha_{sl}\beta_{sl},_{T_1}=-k\alpha_{sl}\{S_{11}^I\alpha_{sl}^2+S_{12}^I\alpha_{s'l'}^2+[R_1^I\cos(2\theta)+R_1^R\sin(2\theta)]\alpha_{s'l'}^2\}\tag{64b}$$

$$\alpha_{s'l'},_{T_1}=-k[R_2^R\cos(2\theta)+R_2^I\sin(2\theta)]\alpha_{sl}^2\alpha_{s'l'}\tag{64c}$$

$$\alpha_{s'l'}\beta_{s'l'},_{T_1}=-k\alpha_{s'l'}\{S_{22}^I\alpha_{s'l'}^2+S_{21}^I\alpha_{sl}^2+[R_2^I\cos(2\theta)-R_2^R\sin(2\theta)]\alpha_{sl}^2\}\tag{64d}$$

where S_i^{I} (i=11, 12, 21, 22) is imaginary part of the S_i (i=11, 12, 21, 22), R_i^{R} and R_i^{I} (i=1, 2) are real part and imaginary part of the R_i (i=1, 2), respectively, and $\theta=\beta_{s'l'}-\beta_{sl}+\sigma T_1$.

There are three possibilities: (1) $\alpha_{sl}=0$ and $\alpha_{s'l'}\neq0$; (2) $\alpha_{s'l'}=0$ and $\alpha_{sl}\neq0$; and (3) $\alpha_{sl}\neq0$ and $\alpha_{s'l'}\neq0$.

In the first uncoupled case, we can similarly obtain the $s'l'$th nonlinear frequency:

$$\Omega_{s'l'}=\omega_{s'l'}-\varepsilon kS_{22}^I\alpha_{s'l'0}^2\tag{65}$$

In the second uncoupled case, we can similarly obtain the slth nonlinear frequency:

$$\Omega_{sl}=\omega_{sl}-\varepsilon kS_{11}^I\alpha_{sl0}^2\tag{66}$$

In the third coupled case, differentiating θ once with respect to T_1 and using Eqs. (64b) and (64d), we obtain

$$\theta_{,T_1} = \sigma + k\left[(S^I_{11}-S^I_{21})\alpha^2_{sl} + (S^I_{12}-S^I_{22})\alpha^2_{s'l'} - \cos(2\theta)(R^I_2\alpha^2_{sl}-R^I_1\alpha^2_{s'l'}) + \sin(2\theta)(R^R_2\alpha^2_{sl}-R^R_1\alpha^2_{s'l'})\right] \tag{67}$$

For steady-state solutions, the amplitude α_{sl} and $\alpha_{s'l'}$ and the new phase θ angle should be constant. Then we obtain

$$-k[R^R_1\cos(2\theta) - R^I_1\sin(2\theta)]\alpha_{sl}\alpha^2_{s'l'} = 0 \tag{68a}$$

$$-k[R^R_2\cos(2\theta) + R^I_2\sin(2\theta)]\alpha^2_{sl}\alpha_{s'l'} = 0 \tag{68b}$$

$$\begin{aligned}\sigma + k\big[(S^I_{11}-S^I_{21})\alpha^2_{sl} + (S^I_{12}-S^I_{22})\alpha^2_{s'l'} - \cos(2\theta)(R^I_2\alpha^2_{sl}-R^I_1\alpha^2_{s'l'})\\ + \sin(2\theta)(R^R_2\alpha^2_{sl}-R^R_1\alpha^2_{s'l'})\big] = 0\end{aligned} \tag{68c}$$

Using Eqs. (68a) and (68b), we obtain

$$\frac{\sin(2\theta)}{\cos(2\theta)} = \frac{R^R_1}{R^I_1} = -\frac{R^R_2}{R^I_2} = C \tag{69}$$

Substituting Eq. (69) into Eq. (68) and eliminating θ yields

$$\frac{\sigma}{\alpha^2_{s'l'}} = -k\left[(S^I_{11}-S^I_{21})\frac{\alpha^2_{sl}}{\alpha^2_{s'l'}} + (S^I_{12}-S^I_{22}) \pm \sqrt{1+C^2}\left|R^I_2\frac{\alpha^2_{sl}}{\alpha^2_{s'l'}} - R^I_1\right|\right] \tag{70}$$

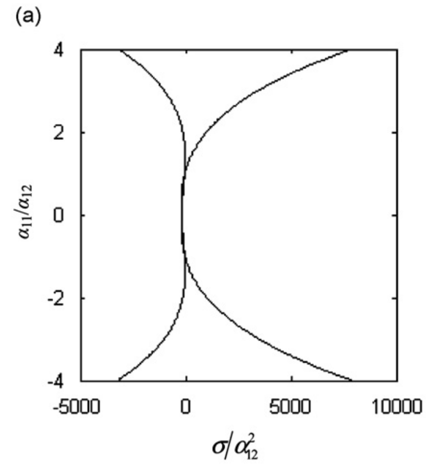

Fig. 6. Variations of the amplitude ratios α_{11}/α_{12} with the differences of the two frequencies σ/α^2_{21} for the case of 1:1 internal resonance between the second and third modes: (a) overview diagram and (b) detail with enlarged scale.

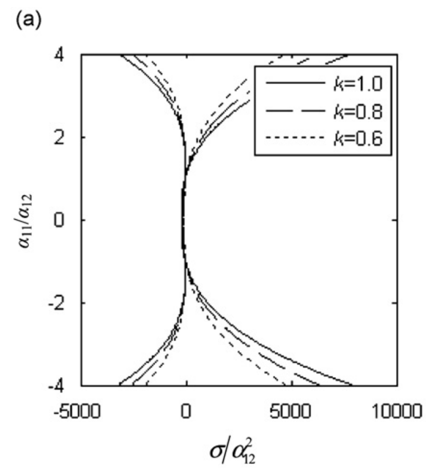

Fig. 7. Variations of the amplitude ratios α_{11}/α_{12} with the differences of the two frequencies σ/α^2_{21} for the case of 1:1 internal resonance between the second and third modes at different nonlinear coefficient: (a) overview diagram and (b) detail with enlarged scale.

1:1 internal resonances between the second and third modes of the in-plane moving plates with k=1 may be activated when γ is near γ=1.8231. Fig. 6 shows variation of the amplitude ratios α_{11}/α_{12} with the differences of the two frequencies σ/α_{12}^2 for this case. It can be found that two solutions exist when $\sigma/\alpha_{12}^2 < -210.8$ or $\sigma/\alpha_{12}^2 > -25.54$, four solutions exist when $\sigma/\alpha_{12}^2 = -210.8$, $\sigma/\alpha_{12}^2 = -56.456$, or $\sigma/\alpha_{12}^2 = -25.54$, six solutions exist when $-210.8 < \sigma/\alpha_{12}^2 < -200.948$, $-200.948 < \sigma/\alpha_{12}^2 < -56.456$, or $-56.456 < \sigma/\alpha_{12}^2 < -25.54$, and five solutions exist when $\sigma/\alpha_{12}^2 = -200.948$.

Fig. 7 shows variation of the amplitude ratios α_{11}/α_{12} with the differences of the two frequencies σ/α_{12}^2 for the case of 1:1 internal resonance between the second and third modes of the in-plane moving plates at different nonlinear coefficients. The dotted lines denote k=0.6, the dashed lines denote k=0.8, and the solid lines denote k=1.0. The relationship between σ/α_{12}^2 and the nonlinear coefficient at a given α_{11}/α_{12} is linear.

7. Numerical confirmations via the differential quadrature scheme

The first four-order dimensionless natural frequencies of the linear generating system and the nonlinear frequency, predicted analytically in the previous section, also can be numerically confirmed. The differential quadrature scheme will be developed here to solve numerically the linear generating system and the primitive system for the two-dimensional full plate model and the one-dimensional reduced plate model.

7.1. Two-dimensional numerical approach

Consider the unit square domain$0 \le x \le 1$, $0 \le y \le 1$ of the in-plane moving rectangular plate. The number of sampling points are N_x and N_y in the x and y directions, respectively. The partial derivatives of $w(x, y)$ at any sampling point (x_i, y_i) as the weighted linear sum of the function w_{ij} values at all the sampling points chosen in the solution domain of spatial variable. The partial derivative of the rth order with respect to x, the partial derivative of the sth order with respect to y, the mixed partial derivative of the sth order with respect to y and the rth order with respect to x are described as follows, respectively [38]:

$$\frac{\partial^r w(x_i,y_j)}{\partial x^r} = \sum_{k=1}^{N_x} A_{ik}^{(r)} w_{kj},\quad \frac{\partial^s w(x_i,y_j)}{\partial y^s} = \sum_{l=1}^{N_y} B_{jl}^{(s)} w_{il},\quad \frac{\partial^{r+s} w(x_i,y_j)}{\partial x^r \partial y^s} = \sum_{k=1}^{N_x} A_{ik}^{(r)} \sum_{l=1}^{N_y} B_{jl}^{(s)} w_{kl}$$
$$(i=1,2,\ldots,N_x,\quad k=1,2,\ldots,N_x-1,\quad j=1,2,\ldots,N_y,\quad l=1,2,\ldots,N_y-1) \tag{71}$$

where the weight coefficients with the recurrence relationship are

$$A_{ik}^{(1)} = \begin{cases} \prod\limits_{\mu=1,\mu\neq i}^{N_x} (x_i - x_\mu) \Big/ \left[(x_i - x_k) \prod\limits_{\mu=1,\mu\neq k}^{N_x} (x_k - x_\mu) \right] (i,k=1,2,\ldots,N_x,\quad i\neq k) \\ \sum\limits_{\mu=1,\mu\neq i}^{N_x} \dfrac{1}{x_i - x_\mu} \quad (i=1,2,\ldots,N_x,\quad i=k) \end{cases} \tag{72}$$

$$B_{jl}^{(1)} = \begin{cases} \prod\limits_{\mu=1,\mu\neq j}^{N_y} (y_j - y_\mu) \Big/ \left[(y_j - y_l) \prod\limits_{\mu=1,\mu\neq l}^{N_y} (y_l - y_\mu) \right] (j,l=1,2,\ldots,N_y,\quad j\neq l) \\ \sum\limits_{\mu=1,\mu\neq j}^{N_y} \dfrac{1}{y_j - y_\mu} \quad (j=1,2,\ldots,N_y,\quad j=l) \end{cases} \tag{73}$$

and in case of r=2, 3, …, N_x-1, s=2, 3, …, N_y-1,

$$A_{ik}^{(r)} = \begin{cases} r\left(A_{ii}^{(r-1)} A_{ik}^{(1)} - \dfrac{A_{ik}^{(r-1)}}{x_i - x_k} \right) \quad (i,k=1,2,\ldots,N_x,\quad i\neq k) \\ -\sum\limits_{\mu=1,\mu\neq i}^{N_x} A_{i\mu}^{(r)} \quad (i=1,2,\ldots,N_x,\quad i=k) \end{cases} \tag{74}$$

$$B_{jl}^{(s)} = \begin{cases} s\left(B_{jj}^{(s-1)} B_{jl}^{(1)} - \dfrac{B_{jl}^{(s-1)}}{y_j - y_l} \right) (j,l=1,2,\ldots,N_y,\quad j\neq l) \\ -\sum\limits_{\mu=1,\mu\neq j}^{N_y} B_{j\mu}^{(s)} \quad (j=1,2,\ldots,N_y,\quad j=l) \end{cases} \tag{75}$$

Introduce N_x and N_y sampling points as

$$\left.\begin{aligned} x_1=0,\quad x_{N_x}=1,\quad x_i=\frac{1}{2}\left[1-\cos\left(\frac{i-1}{N_x-1}\pi\right)\right](i=2,3,\ldots,N_x-1),\\ y_1=0,\quad y_{N_y}=1,\quad y_j=\frac{1}{2}\left[1-\cos\left(\frac{j-1}{N_y-1}\pi\right)\right](j=2,3,\ldots,N_y-1)\end{aligned}\right\} \tag{76}$$

Following the above definitions, the DQ analogue of the linear generating system can be obtained as

$$\begin{aligned}\ddot{w}_{0ij}+2\gamma\sum_{k=2}^{N_x-1}A_{ik}^{(1)}\dot{w}_{0kj}+(\gamma^2-1)\sum_{k=2}^{N_x-1}\tilde{A}_{ik}^{(2)}w_{0kj}+\zeta\left(\sum_{k=2}^{N_x-1}\tilde{A}_{ik}^{(4)}w_{0kj}+\xi^4\sum_{l=2}^{N_y-1}\tilde{B}_{jl}^{(4)}w_{0il}\right.\\ \left.+2\xi^2\sum_{k=2}^{N_x-1}\tilde{A}_{ik}^{(2)}\sum_{l=2}^{N_y-1}\tilde{B}_{jl}^{(2)}w_{0kl}\right)=0\quad(i=2,3,\ldots,N_x-1,\,j=2,3,\ldots,N_y-1)\end{aligned} \tag{77}$$

where $\tilde{A}_{ik}^{(r)}$ and $\tilde{B}_{jl}^{(s)}$ are the modified weighting coefficient matrices. In Eq. (77) the simple supported boundary conditions are built in.

7.2. One-dimensional numerical approach

The solution to Eq. (11) can be assumed as

$$w(x,y,t)=\sum_{m=1}^{\infty}\phi(x,t)\varphi_m(y)+cc \tag{78}$$

where $\varphi_m(y)=\sin(m\pi y)$ is the mth modal function in the y direction of the system and cc stands for complex conjugate of the proceeding terms.

Substitution of Eq. (78) into (11), multiplying $\varphi_m(y)$, and integrating from $y=0$ to 1, then applying the orthogonal condition, the DQ analogue of the primitive system can be obtained as

$$\begin{aligned}\ddot{\phi}_i=&-2\gamma\sum_{k=2}^{N_x-1}A_{ik}^{(1)}\dot{\phi}_k-(\gamma^2-1)\sum_{k=2}^{N_x-1}\tilde{A}_{ik}^{(2)}\phi_k-\zeta\left(\sum_{k=2}^{N_x-1}\tilde{A}_{ik}^{(4)}\phi_k-2m^2\pi^2\xi^2\sum_{k=2}^{N_x-1}\tilde{A}_{ik}^{(2)}\phi_k\right.\\ &\left.+m^4\pi^4\xi^4\phi_i\right)+\frac{3}{2}\varepsilon k\zeta\left[3\left(\sum_{k=2}^{N_x-1}\tilde{A}_{ik}^{(2)}\phi_k-\mu m^2\pi^2\xi^2\phi_i\right)\left(\sum_{k=2}^{N_x-1}A_{ik}^{(1)}\phi_k\right)^2+m^2\pi^2\xi^2\left(\mu\sum_{k=2}^{N_x-1}\tilde{A}_{ik}^{(2)}\phi_k\right.\right.\\ &\left.\left.-m^2\pi^2\xi^2\phi_i\right)\phi_i^2+2m^2\pi^2\xi^2(1-\mu)\left(\sum_{k=2}^{N_x-1}A_{ik}^{(1)}\phi_k\right)^2\phi_i\right]+o(\varepsilon)\quad(i=2,3,\ldots,N_x-1)\end{aligned} \tag{79}$$

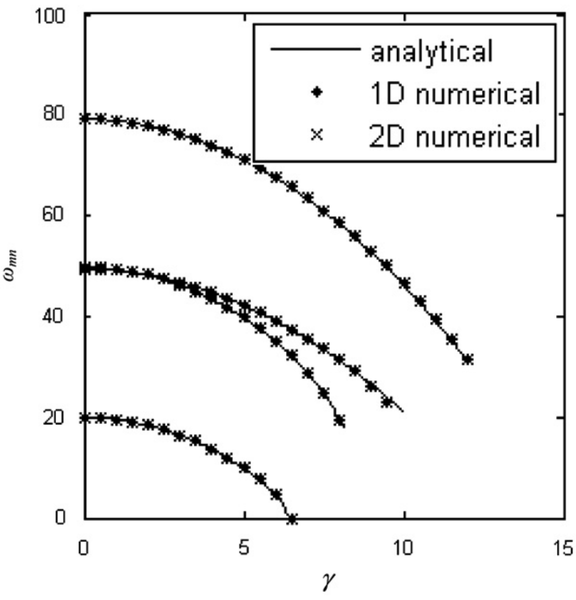

Fig. 8. The comparison of the first four-order dimensionless natural frequencies.

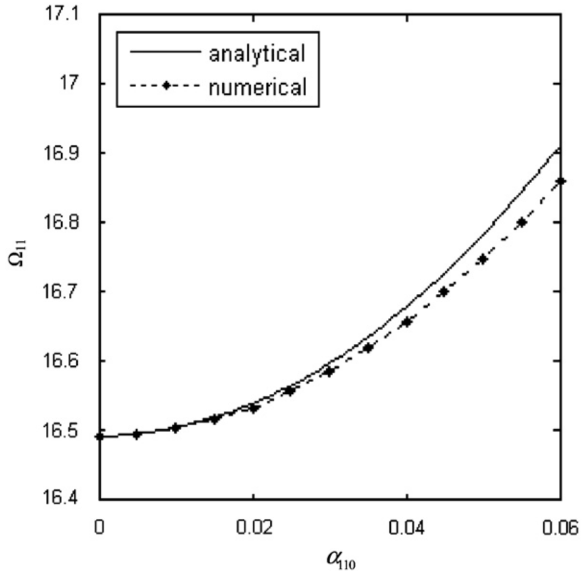

Fig. 9. Comparison of analytical and numerical nonlinear frequencies for different nonlinear coefficients *k*.

7.3. Numerical demonstrations

In this paper, we choose sampling points N_x=N_y=9 in two-dimensional approach and N_x=9 in one-dimensional approach. Fig. 8 presents the comparison for the variation of the first four-order dimensionless natural frequencies of the linear generating system with dimensionless in-plane moving speed for ζ=1, ξ=1. It can be observably found that the one-dimensional numerical approach and two-dimensional numerical approach have a good agreement with the complex mode approach.

It can be found that the results of the one-dimensional approach, the two-dimensional approach and the complex mode approach the not only yield the qualitatively same results, but also the quantitatively close. The three approaches have the same qualitative results. On the other hand, the two-dimensional approach spends much more computationally time than the one-dimensional approach, so we just investigate the comparison of analytical and numerical results by the one-dimensional approach in the following example.

Fig. 9 shows the comparison of the analytical and numerical nonlinear frequencies for the first mode for γ=3.0 and k=1. The amplitude for the first mode is α_{110}=$w/2|\psi_{s'l'}|$. The one-dimensional approach yields the quantitatively close results, while there is a small quantitative difference between the analytical and the one-dimensional numerical results.

8. Conclusions

In this investigation, the nonlinear free transverse vibration of an in-plane moving plate with constant speed is investigated. The governing equation with the boundary conditions is derived from the Hamilton principle and the Hooke law. The method of multiple scales is employed to analyze the governing equation to yield the following conclusions. At last, the differential quadrature schemes are developed for the one-dimensional full plate model and the two-dimensional reduced plate model. The schemes are applied to solve numerically the linear generating systems and the nonlinear governing equations.

In the multi-scale analysis on the case without internal resonances, it is found that, if two modes are considered, a response in one of them must be zero. Therefore, it is valid to investigate no internal resonance cases without the coupling among modes. The relationship between nonlinear frequencies and amplitudes of the in-plane moving plate is showed at different in-plane moving speeds for the first two modes. Both modes predict the same trends of the nonlinear free vibration frequencies varying with the initial amplitudes and the in-plane moving speeds. The nonlinear frequencies increase with the increasing the initial amplitudes. When the initial amplitude is zero, the results reduce the frequencies of the linear generating system. The larger in-plane moving speed leads to the smaller frequencies and the rapider increase of the frequencies with the initial amplitudes, especially, it increases rapidly when the in-plane moving speed is near the critical in-plane moving speed. Besides, the effects of are more significant in the higher order mode. The nonlinear frequencies involving higher order modes are more sensitive to the initial amplitudes, the in-plane moving speeds, and the nonlinear coefficients.

In the multi-scale analysis on the case with internal resonances, the 3:1 internal resonances between the first and second modes and between the third and fourth modes, as well as 1:1 the second and third modes are detected. The relationship between nonlinear frequencies and amplitudes of the in-plane moving plate is demonstrated to be the same as those in the no internal resonance cases. However, the amplitude ratios vary with the detuning parameters. The variations are showed at different nonlinear coefficients. It is found that the relationship between the detuning parameters and the nonlinear coefficients at a given amplitude ratio is linear.

The first four-order linearized natural frequencies are confirmed by the one-dimensional numerical approach and two-dimensional numerical approach. They have a good agreement with the complex mode approach. The three approaches not only yield the qualitatively same results, but also the quantitatively close. They decrease with the increasing in-plane moving speeds.

The comparison of the analytical and numerical nonlinear frequencies for the first mode is also demonstrated. Though the one-dimensional approach yields the quantitatively close results, there is a small quantitative difference between them.

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